ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY

APPLIED MATHEMATICS PROGRAM

Course Outline

Course Title:	Applied Mathematics I	**Lecture Hours:**	3
Course Code:	Math 1101	**Tutorial Hours:**	3
Target Group:	Pre - Engineering Students	**Instructor's Name:**	

Course Content

1. **Vectors**
   - 1.1 Definition of vectors
   - 1.2 Vector operations(addition & scalar multiplication)
   - 1.3 Norm of a vector(with application examples)
   - 1.4 Scalar product (and application examples like work done)
   - 1.5 Projections of vectors
   - 1.6 Cross product (and application examples like torque)
   - 1.7 Applications of cross product(area of //gm & triangle, volume of parallelepiped)
   - 1.8 Equations of lines and planes in space
     - 1.8.1 Distance from a point to a line; distance from a point to a plane
2. **Matrices and Determinants**
   - 2.1 Definition of matrix and types of matrices(some special matrices)
   - 2.2 Operations and properties of matrices; transpose of a matrix(with applied examples)
   - 2.3 Elementary row operations
   - 2.4 Reduced row – echelon form of a matrix; rank of a matrix
   - 2.5 Inverse of a matrix and its properties
   - 2.6 Determinant of a matrix and its properties(with applied examples)
   - 2.7 Solving systems of linear equations
     - 2.7.1 Cramer's rule(with applied examples)
     - 2.7.2 Gaussian's method(with applied examples)
     - 2.7.3 Inverse matrix method(with applied examples)

Note: Mid – term exam covers chapter 1 and 2.

3. **Limits and Continuity**
   - 3.1 Definitions of limits(Intuitive approach)
   - 3.2 Basic limits theorems; One - sided limits
   - 3.3 Infinite limits, limits at infinity, and asymptotes(vertical & horizontal)
   - 3.4 $\varepsilon - \delta$ definition of limits(formal/precise definition)
   - 3.5 Continuity; One - sided continuity

3.6 Intermediate value theorem & its applications

4. **The Derivative and Its Applications**
   - 4.1 Definition of derivatives(with some applied examples); Basic rules
   - 4.2 Derivatives of inverse functions
     - 4.2.1 Brief revision on inverse functions
     - 4.2.2 Inverse trigonometric functions and their derivatives
     - 4.2.3 Hyperbolic and inverse hyperbolic functions and their derivatives
   - 4.3 Higher order derivatives(and some applied examples like acceleration)
   - 4.4 Implicit differentiation
   - **4.5 Applications of differentiation**
     - 4.5.1 Related rate(with applied examples)
     - 4.5.2 Extrema(max – min value) of a function(with applied examples)
     - 4.5.3 First and second derivative tests(with applied examples)
     - 4.5.4 Concavity and inflection points(with applied examples)
     - 4.5.5 Curve sketching
   - 4.6 Indeterminate forms (L'Hopital's rule)
5. **Integration and Its Applications**
   - 5.1 Anti - derivatives; Indefinite integrals
   - 5.2 Techniques of integration
     - 5.2.1 Integration by substitution, and by parts
     - 5.2.2 Integration by partial fraction
     - 5.2.3 Trigonometric integrals
     - 5.2.4 Integration by trigonometric substitution
   - 5.3 Definite integrals; Fundamental Theorem of Calculus
   - 5.4 Improper integrals
   - **5.5 Applications of Integration**
     - 5.5.1 Area (with applied examples)
     - 5.5.2 Volume (with applied examples)
     - 5.5.3 Arc length(with applied examples)
     - 5.5.4 Surface area(with applied examples)

Assessment Methods

- Test and/or quiz (two):- 20%
- Assignments (two):- 20%
- Mid semester examination: -20%
- Final examination: - 40%

Textbook

Robert Ellis and Denny Gulick, Calculus with analytic geometry, 6th edition, Harcourt Brace Jovanovich, publishers, 5th ed, 1993.

References

1. Howard Anton, Calculus with Analytic Function, fifth edition.
2. James Stewart: Calculus Early Transcendentals, 6th edition

CHAPTER-1
Vectors

Defn:

(i) Physical quantities that have only magnitude are called scalars.
   E.g: Length, Mass, Volume, Area, Temperature, etc.

(ii) Physical quantities that have both magnitude & direction are called vectors.
   E.g: Force, Displacement, Velocity, etc.

Notation & Representation of a vector

If A & B are two points in $\mathbb{R}^n$, then the directed line segment from A to B is called a vector $\overrightarrow{AB}$.

- By the vector $\overrightarrow{AB}$, we mean a quantity whose magnitude is the length AB, denoted $\|\overrightarrow{AB}\|$, and whose direction is the direction from A to B. A is called initial point (tail) & B is called terminal point (head).
- Vectors are usually denoted by bold lowercase letters such as $u, v, w, \ldots$.

Note:

(i) Two vectors u & v are said to be equal, denoted $u = v$ iff they have the same magnitude & direction.

(ii) Vectors with initial point at the origin of a coordinate system are called position (located) vectors. We name position vectors by the coordinates of their terminal point.

(iii) If $u = \overrightarrow{AB}$ is a vector in $\mathbb{R}^n$, then there is a position vector $\overrightarrow{OP}$ with O-origin & $P = B - A$ such that $\overrightarrow{AB} = \overrightarrow{OP}$. Thus,
$$\overrightarrow{AB} = \text{Vector } (B - A).$$

E.g: 1) If $u = \overrightarrow{AB}$ is a vector in $\mathbb{R}^2$ with $A = (2,3)$ & $B = (5,7)$, then $u = \overrightarrow{AB} = (3,4) = 3i + 4j$.

2) Let $A = (-1,-2,-3)$ be a point & $w = i - j + 2\hat{k}$ be a vector. Find a point B such that $\overrightarrow{AB}$ & w are equal.

Vector Operations

(I) Vector Addition

Let u & v be two vectors in $\mathbb{R}^n$.

- By connecting head of u to tail of v, $u+v$ is the vector from initial point of u to terminal point of v. This is Triangular Law of Vector Addition.
- By constructing a parallelogram with adjacent sides u & v (By drawing u & v from common initial point), $u+v$ is the diagonal of the parallelogram beginning from initial point of u (and v). This is Parallelogram Law of Vector Addition. (Identity element / Zero vector - Additive Identity $(-u)$ Left & Right >>
- If $u=(u_1, u_2, \dots, u_n)$ & $v=(v_1, v_2, \dots, v_n)$, then $u+v=(u_1+v_1, u_2+v_2, \dots, u_n+v_n)$. This is Component Wise Vector Addition.

The Zero-Vector / Null-vector, denoted 0, is a vector with zero magnitude & any direction.

(II) Vector Subtraction

Let u & v be two vectors in $\mathbb{R}^n$. Then $u-v$ is the vector $u+(-v)$.

- By drawing u & v from a common initial point, $u-v$ is the vector from terminal point of v to terminal point of u.
- If $u=(u_1, u_2, \dots, u_n)$ & $v=(v_1, v_2, \dots, v_n)$, then $u-v=(u_1-v_1, u_2-v_2, \dots, u_n-v_n)$.

(III) Scalar Multiplication

Scalar multiple of vector $u=(u_1, u_2, \dots, u_n)$ by $k \in \mathbb{R}$ is defined as

$ku = (ku_1, ku_2, \dots, ku_n)$.

<u>Note</u>: Two non-zero vectors u & v in $\mathbb{R}^n$ are said to be parallel, denoted $u \parallel v$ iff $u = kv$ for some $k \in \mathbb{R}$, $k \neq 0$.

If $k>0$, u & v are in the same direction.

If $k<0$, u & v are in opposite direction.

E.g.:

1) Let $u=(1,-2,3)$ & $v=(2,3,1)$ be vectors in $\mathbb{R}^3$. Find vector w in $\mathbb{R}^3$ such that $v+3w=-2u$.

2) Let $u=\vec{AB}$ & $v=\vec{PQ}$ where $A=(1,2,-5)$, $B=(3,6,5)$, $P=(1,4,2)$ & $Q=(0,2,-3)$. Determine whether u & v are parallel, & are in the same direction or not.

Norm of a vector

For vector $u = (u_1, u_2, \ldots, u_n)$ in $\mathbb{R}^n$, norm (length or magnitude) of u, denoted $\|u\|$, is defined as $\|u\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$.

- A vector with norm 1 is called a unit vector.
  - In $\mathbb{R}^2$, standard unit vectors are $(1,0)$ & $(0,1)$, denoted i & j respectively.
  - In $\mathbb{R}^3$, standard unit vectors are $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$, denoted i, j & k resp.

Note:

(i) For vector u in $\mathbb{R}^n$,

(a) $\|u\| \geq 0$, & $\|u\| = 0$ iff $u = 0$

(b) $\|ku\| = |k|\,\|u\|$ for any $k \in \mathbb{R}$

(c) $\|-u\| = \|u\|$.

(ii) For non-zero vector u in $\mathbb{R}^n$, $\frac{u}{\|u\|}$ (or $\frac{1}{\|u\|}u$) is a unit vector in the direction of u.

(iii) For points A & B in $\mathbb{R}^n$, distance d between A & B is

$$d = \|A - B\| = \|B - A\|.$$

e.g.:

1) Find $\|u\|$ & $\|v\|$, for vectors $u = (-2, 1, -2)$ & $v = (2, 0, 3)$.
2) Find $\|w\|$, where $w = \overrightarrow{AB}$ with $A = (1, 2)$ & $B = (4, 6)$.
3) Find value(s) of $k \in \mathbb{R}$ so that $u = (k, 2k, 2k)$ is a unit vector. $1/3, -1/3$
4) Find a unit vector in the direction of vector $(0, 4, -3)$, & a unit vector in the direction opposite to vector $(1, -2, 5)$.
5) Find vector u in $\mathbb{R}^3$ with magnitude 9 in the direction of vector $V = (2, -1, 2)$.
6) Find the distance between two points $A = (3, 1, 2)$ & $B = (-2, 2, -1)$ in $\mathbb{R}^3$.

$$\vec{u} = \frac{\vec{V}}{\|\vec{V}\|}$$

(IV) Vector Multiplication

(A) Dot /scalar/ product

For vectors $u = (u_1, u_2, \ldots, u_n)$ & $v = (v_1, v_2, \ldots, v_n)$, dot product of u & v, denoted $u \cdot v$, is defined as

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

• Properties:

Let u, v & w be vectors in $\mathbb{R}^n$ & $\alpha, \beta \in \mathbb{R}$.

(i) (a) $u \cdot v = v \cdot u$

(b) $u \cdot (v + w) = u \cdot v + u \cdot w$

(ii) (a) $\alpha(u \cdot v) = (\alpha u) \cdot v = u \cdot (\alpha v)$

(b) $u \cdot (\alpha v + \beta w) = \alpha(u \cdot v) + \beta(u \cdot w)$

(iii) $u \cdot u \geq 0$, & $u \cdot u = 0$ iff $u = 0$.

(iv) $\|u\| = \sqrt{u \cdot u}$

(v) $0 \cdot u = 0$

cauchy-schwarz inequality

(vi) $|u \cdot v| \leq \|u\| \|v\|$, &

$|u \cdot v| = \|u\| \|v\|$ iff $u = kv$, where $k \in \mathbb{R}$.

(vii) $\|u + v\| \leq \|u\| + \|v\|$.

Minkowski inequality (triangle)

Note:

(i) If θ is an included angle between vectors u & v, then

$$u \cdot v = \|u\| \|v\| \cos\theta.$$

(ii) Two non-zero vectors u & v in $\mathbb{R}^n$ are said to be orthogonal (perpendicular), denoted $u \perp v$ iff $u \cdot v = 0$.

e.g.:

1) For any two vectors u & v, show that $\|u+v\|^2 - \|u-v\|^2 = 4(u \cdot v)$.

2) Find the angle between vectors $u = (3, 2)$ & $v = (-4, 6)$.

3) Let angle between vectors u & v be $120°$, $\|u\| = 3$ & $\|v\| = 4$. Find $\|u + 2v\|$.

4) For vectors u, v, w in $\mathbb{R}^n$, let $u + v + w = 0$, $\|u\| = 3$, $\|v\| = 5$ & $\|w\| = 7$. Find the angle between u & v.

5) Find $\alpha \in \mathbb{R}$ so that vectors $(\alpha, -3, 1)$ & $(\alpha, \alpha, 2)$ are perpendicular.

6) Show that diagonals of a parallelogram are orthogonal iff the parallelogram is a rhombus.

Projection of a vector along another Vector

For two non-zero Vectors u & v in $\mathbb{R}^n$ with common initial point, the vector obtained by dropping the head of u onto v is called the projection of u onto (along) v, denoted by Proj_v^u (or simply Pr_v^u).

And $\text{Proj}_v^u = \left(\frac{u \cdot v}{\|v\|^2}\right) v$.

Proj_v^u is in the same direction of v iff included angle θ is less than $\pi/2$, i.e. $0 \le \theta < \pi/2$.

Proj_v^u is in opposite direction to v iff $\frac{\pi}{2} < \theta \le \pi$.

$\text{Proj}_v^u = 0$ iff $\theta = \pi/2$.

E.g:

1) For Vectors $u = (2,3,1)$, $v = (0,-3,4)$ & $w = (3,2,-1)$, find

(a) Proj_v^u (b) Proj_{-u}^v (c) Proj_{u+w}^v (d) Proj_{u-v}^w.

2) Vectors u & v with magnitude 3 & 2 respectively, form an included angle $\pi/3$. Find

(a) $\|\text{Proj}_v^u\|$ (b) $\|\text{Proj}_u^v\|$

3) For Vectors $u = (3,-1,-2)$ & $v = (2,-3,1)$, find component Vector of u orthogonal to v.

4) Find a unit Vector orthogonal to vector $u = (2,-1)$.

Read directional angles & directional cosines of a vector in $\mathbb{R}^3$.

(B) Cross/Vector/ product

For Vectors $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$, cross product of u & v, denoted $u \times v$, is a vector in $\mathbb{R}^3$ defined as

$$u \times v = (u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1).$$

E.g: For Vectors $u = (-2,-3,-1)$, $v = (1,3,2)$, $a = (2,1,1)$ & $b = (4,2,2)$, find

(a) $u \times v$ (b) $v \times u$ (c) $a \times b$ (d) $b \times a$.

• Properties:

Let u, v, w be vectors in $\mathbb{R}^3$ & $\alpha \in \mathbb{R}$.

(i) (a) $u \times v = -(v \times u)$ - not commutative || opposite dirn in equal mag -

(b) $u \times (v+w) = (u \times v) + (u \times w)$ & $(u+v) \times w = (u \times w) + (v \times w)$.

(c) $u \times (v \times w) \neq (u \times v) \times w$. not Associative

(ii) $\alpha (u \times v) = (\alpha u) \times v = u \times (\alpha v)$ either of them

(iii) $u \times v = 0$ iff $u // v$.

(iv) $u \times 0 = 0$

(v) $u \cdot (u \times v) = 0$ & $v \cdot (u \times v) = 0$

(vi) $\|u \times v\|^2 = \|u\|^2 \|v\|^2 - (u \cdot v)^2$ ---- (Lagrange's Identity).

Note:

(i) If θ is an included angle between vectors u & v in $\mathbb{R}^3$, then

$$\|u \times v\| = \|u\| \, \|v\| \sin \theta$$

(ii) For non-zero vectors u, v, w in $\mathbb{R}^3$,

(a) area of a parallelogram with adjacent sides u & v is

$$\text{Area} = \|u \times v\|.$$

(b) Surface area of a parallelopiped determined by u, v & w is

$$\text{Surface area} = 2\left(\|u \times v\| + \|u \times w\| + \|v \times w\|\right).$$

e.g.:

1) For non-zero vectors u, v, w in $\mathbb{R}^3$, find $(u \times (v+w)) + (v \times (w+u)) + (w \times (u+v))$.

2) For vectors u, v, w in $\mathbb{R}^3$, let $u+v+w=0$. Show that $u \times v = v \times w = w \times u$.

3) Find a unit vector orthogonal to both vectors $u = (-2,-3,-1)$ & $v = (1,3,2)$.

4) Find a unit vector orthogonal to vector $u = (2,-2,1)$.

5) Find area of a parallelogram determined by vectors $u = (2,1,3)$ & $v = (1,-1,2)$.

6) Find surface area of a parallelopiped generated ~~determined~~ by vectors $u = (2,1,-1)$, $v = (3,2,1)$ & $w = (1,1,2)$.

7) For points $A = (3,-2,1)$, $B = (2,3,1)$ & $C = (2,1,0)$, find area of triangle ABC.

Triple Scalar & Triple Vector Products

For vectors u, v, w in $\mathbb{R}^3$, products

(i) $u\cdot(v\times w)$, $w\cdot(u\times v)$, $v\cdot(u\times w)$, --- are called triple scalar products.

(ii) $u\times(v\times w)$, $(u\times v)\times w$, $v\times(u\times w)$, --- are called triple vector products.

Note:

(i) For vectors u, v, w in $\mathbb{R}^3$,

(a) $|u\cdot(v\times w)| = |w\cdot(u\times v)| = |v\cdot(u\times w)|$.

(b) $u\times(v\times w) = (u\cdot w)v - (u\cdot v)w$ &

$$(u\times v)\times w = (w\cdot u)v - (w\cdot v)u.$$

(ii) Volume of a parallelopiped determined by non-zero vectors u, v, w in $\mathbb{R}^3$ is absolute value of triple scalar product of u, v & w.

e.g.:

1) Let u, v, w be vectors in $\mathbb{R}^3$.

(a) Show that $(u\times(v\times w)) + (v\times(w\times u)) + (w\times(u\times v)) = 0$

(b) If $u \perp w$, show that $u\times(v\times w) \perp (u\times v)\times w$.

2) Find volume of a parallelopiped determined by vectors

$u = (-1, 1, 1)$, $v = (0, 2, -1)$ & $w = (2, -1, -1)$.

(Find volume of a ||piped with three adjacent edges formed by vectors u, v & w)

Parametric Equation of lines & planes / Parametric equation of a line /

Defn: Let line ℓ pass through point P & parallel to vector u. Then, the equation $X = P + tu;\ t \in \mathbb{R}$, is called parametric equation of line ℓ.

(u is called the direction vector of line ℓ & t is the parameter.)

Note:

(I) Relation between parametric & cartesian equations of line ℓ in $\mathbb{R}^2$:

* Let parametric equation of line ℓ be $(x,y) = (a,b) + t(c,d)$. Then,

(a) if $c \neq 0$ & $d \neq 0$, then cartesian equation of line ℓ is

$$y = \frac{d}{c}x + \left(b - \frac{da}{c}\right).$$

(b) if $c = 0$ & $d \neq 0$, then cartesian equation of line ℓ is $x = a$.

(i.e. a vertical line through $(a,0)$).

(c) if $c \neq 0$ & $d = 0$, then cartesian equation of line ℓ is $y = b$.

(i.e. a horizontal line through $(0,b)$).

* Let cartesian equation of line ℓ be $y = ax + b$. Then, parametric equation of line ℓ is $(x,y) = (0,b) + t(1,a);\ t \in \mathbb{R}$.

(II) Distance between a point & a line in $\mathbb{R}^3$:

Let line ℓ be with direction vector u & point Q not on ℓ. Then, distance d between Q & ℓ is

$$d = \frac{\| u \times \vec{PQ} \|}{\| u \|} \quad \text{for any point } P \text{ on } \ell.$$

Eg:

1) Find parametric equation of a line passing through

(a) point $(1,2,3)$ & parallel to vector $(1,1,1)$.

(b) points $(2,-1,0)$ & $(4,5,3)$.

2) Find cartesian equation of a line with parametric equation $(x,y) = (2,4) + t(-3,1);\ t \in \mathbb{R}$.

3) Find parametric equation of a line with cartesian equation $3x + 2y = 4$.

4) Find the distance between point $(2,1,-1)$ & a line with parametric equation $(x,y,z) = (0,1,-5) + t(3,2,-1)$.

5) Find the distance between lines with parametric equations $(x,y,z) = (1,-1,2) + t(2,-1,-2)$ & $(x,y,z) = (0,2,3) + t(2,-1,-2);\ t \in \mathbb{R}$.

6) Find a point which lies two-third of the way from point $(1,2,1)$ to point $(4,5,3)$.

Parametric Equation of plane

Def: Let plane π pass through non-collinear points P, Q & R. Then, the equation $X = P + t(\vec{PQ}) + s(\vec{PR})$; $t, s \in \mathbb{R}$, is called parametric equation of plane π.

Note:

(I) plane π passing through point P & perpendicular to vector N is the collection of all points X with $\vec{PX}$ perpendicular to N.
N is called the normal vector of plane π.

(II) Cartesian equation of a plane passing through point (x_0, y_0, z_0) & normal vector (a, b, c) is $\boxed{ax + by + cz = d}$ where $d = ax_0 + by_0 + cz_0$.
Cartesian eqn

(III) Let planes π_1 & π_2 be with normal vectors N_1 & N_2 respectively. Then,

(a) $\pi_1 \parallel \pi_2$ iff $N_1 \parallel N_2$, and $\pi_1 \perp \pi_2$ iff $N_1 \perp N_2$.

(b) Angle between π_1 & π_2 = Angle between N_1 & N_2.

(IV) Let plane π be with normal N & point Q not on π. Then, distance d between Q & π is

$$d = \frac{|N \cdot \vec{PQ}|}{\|N\|} \quad \text{for any point } P \text{ on } \pi.$$

Ex:

1) Find Cartesian & parametric equations of a plane
(a) Containing points $(2, 0, 1)$, $(0, 3, -1)$ & $(3, -1, 1)$.
(b) with normal $(2, 3, -1)$ & passing through point $(2, 1, -1)$.

2) Find the cosine of the angle between planes $2x + 4y - z = 0$ & $x - y + 2z = 2$.

3) Find the distance between point $(1, 3, 5)$, and plane with normal $(-1, 1, -1)$ & passing through point $(-1, 1, 7)$.

4) Find intersection point of a line with equation $(x, y, z) = (-1, -3, 0) + t(2, 3, -1)$ & a plane with equation $3x - 2y + 4z = -1$.

5) Find parametric equation of line l which is intersection of two planes $\pi_1: 2x - 3y + 4z = 2$ & $\pi_2: x - z = 1$.

CHAPTER - THREE
MATRICES

Defn: A matrix is a rectangular array of numbers or variables arranged in rows & columns. Succession

- A matrix with m rows & n columns is said to be a matrix of order $m \times n$ (or an $m \times n$ matrix).

Matrices are usually denoted by capital letters such as A, B, C, ...

Note:

Let matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$

(i) A is an $m \times n$ matrix, denoted as $A = (a_{ij})_{m \times n}$.

(ii) a_{ij}; $i = 1, 2, \cdots, m$ & $j = 1, 2, \cdots, n$, is the element appearing in the i^{th} row & j^{th} column of A, called $(ij)^{th}$ entry / $(ij)^{th}$ component / of A.

(iii) The i^{th} row of A is denoted by A_i, $i = 1, 2, \cdots, m$ & the j^{th} column of A is denoted by A^j, $j = 1, 2, \cdots, n$. That is,

$A_i = (a_{i1} \; a_{i2} \; \cdots \; a_{in})$ & $A^j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$.

Thus, $A = \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix}$ or $A = (A^1 \; A^2 \; \cdots \; A^n)$.

E.g.:

1) Let $A = \begin{pmatrix} 1 & 2 & 3 & 7 \\ 8 & 4 & 9 & 6 \\ 5 & 8 & 7 & 1 \end{pmatrix}$. Find

(a) order (size) of A.

(b) elements a_{23}, a_{32}, a_{24}, a_{42} & a_{33}.

(c) A_{3} row & A^3 = column $(~~3\;9\;7~~)$ $\begin{pmatrix} 3 \\ 9 \\ 7 \end{pmatrix}$

2) Construct a (4×3) matrix $A = (a_{ij})$, where $a_{ij} = i + j$.

Special Matrices

- Row matrix (vector) :- A matrix with one row.
- Column Matrix (vector) :- A matrix with one column.
- Zero / Null / matrix :- A matrix where its elements are all zero, denoted by O.
- Square matrix :- A matrix where no. of rows = no. of columns.
- Diagonal matrix :- A square matrix where all except possibly the diagonal elements are zero.
- Scalar matrix :- A diagonal matrix where the diagonal elements are equal.
- Identity / unit / matrix :- A diagonal matrix where the diagonal elements are all 1.

- Triangular matrix:- A matrix which is either an upper or a lower triangular matrix.
  - • upper triangular matrix :- A square matrix where elements below the diagonal are all zero.
  - • Lower triangular matrix :- A square matrix where elements above the diagonal are all zero.

<u>equality of matrices</u>

Matrix A = matrix B iff A & B have the same order, and corresponding elements are equal.

Eg: Find $x, y, z \in \mathbb{R}$ so that $A = B$, where
$$A = \begin{pmatrix} x+y^2 & -1 \\ 2 & 2y-1 \end{pmatrix} \quad \& \quad B = \begin{pmatrix} 3 & 3x-2y \\ 5x-\frac{z}{3} & 3 \end{pmatrix}.$$

Matrix Operations

- <u>Matrix Addition</u>:-
  $A+B$ is the matrix obtained by adding corresponding elements of matrices A & B, and is defined if A & B are of the same order. (closerty
  O-Additive $\quad A + O_{mn} = O_{mn} + A$
- <u>Scalar Multiplication</u>:-
  $\alpha A, \alpha \in \mathbb{R}$ is the matrix obtained from matrix A by multiplying each element of A by α.
  $A+B$ - Add
  $A + -1B$ - scalar Mult $\quad A-B = A + -1B$
- <u>Matrix multiplication</u>:-
  If $A = (a_{ij})_{m\times n}$ & $B = (b_{ij})_{n \times p}$, then $A \cdot B = (c_{ij})_{m\times p}$, where $c_{ij} = A_i \cdot B^j$.
  - • matrix multiplication is
    - not commutative
    - associative
    - distributive over matrix addition.
- • $\alpha(AB) = (\alpha A)B = A(\alpha B)$, where $\alpha \in \mathbb{R}$.
- • If A is a square matrix of order n, $AI_n = A = I_nA$. — Multiplicative Identity
- • If multiplicative inverse A^{-1} of a square matrix A exists, then A is called invertible (non-singular). Otherwise non-invertible (singular).

<u>Eg</u>:

1) Find matrix C, where $A+B = 2C$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$.

2) If $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 3 & 1 & 2 \end{pmatrix}$, find matrix B so that $A+B$ is a
   (a) diagonal matrix (b) scalar matrix (c) Identity matrix.

3) Find $x+y$, where $A = \begin{pmatrix} -1 & 3 \\ -5 & -3 \end{pmatrix}$ is additive inverse of
   $B = \begin{pmatrix} 2x-y & -3 \\ x+y & 2y-3 \end{pmatrix}$.
   $A = -B$
   $-B = A$

4) Find $\boxed{AB}$, AC, BA, $\boxed{BC}$, CA, CB if exist, where
$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}_{2\times 2}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}_{2\times 3}$ & $C = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}_{3\times 3}$

5) Find $(A)^2$ & $(A)^3$, where $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

6) Find multiplicative inverse of A & B if exist, where
$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 4 \\ 2 & 8 \end{pmatrix}$.

Transpose of a matrix

- Transpose of matrix A, denoted A^t (or A^T), is the matrix obtained from A by interchanging its rows with columns.
- Square matrix A of order n is said to be an orthogonal matrix iff $AA^t = I_n = A^tA$.
- For a square matrix A,
  - if $A^t = A$, then A is said to be a symmetric matrix.
  - if $A^t = -A$, then A is said to be a skew symmetric matrix.
- For any square matrix A, $(A + A^t)$ is a symmetric matrix & $(A - A^t)$ is a skew symmetric matrix.
- Since $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$, any square matrix can be written as a sum of symmetric & skew-symmetric matrices.

E.g:

1) Find A^t & B^t, where $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 2 & 1 & -3 & 0 \\ 3 & 2 & 4 & 1 \\ 4 & 0 & 2 & -1 \end{pmatrix}$.

2) Express $C = \begin{pmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ 6 & 4 & -4 \end{pmatrix}$ as a sum of symmetric & skew symmetric matrices.

3) If A symmetric & B skew-symmetric matrices of same order, show that A^2BA^2 is skew-symmetric.

Determinant of a square matrix

- If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then $\det A = |A| = a_{11}a_{22} - a_{12}a_{21}$.
- If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $\det A =$
$$a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$
- If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$, then
$$\det A = a_{11}\begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix} - a_{12}\begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix} + a_{13}\begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix} - a_{14}\begin{vmatrix} \vdots & \vdots & \vdots \end{vmatrix}.$$

- For square matrix A,
  - Minor of A, denoted M_{ij}, is determinant of the matrix after i^{th} row & j^{th} column have been deleted from A.
  - Cofactor of A, denoted A_{ij} is $(-1)^{i+j} M_{ij}$.
  - Cofactor matrix of A is matrix of Cofactors.
  - Adjoint matrix of A, denoted $\operatorname{adj} A$ is $(\text{Cofactor matrix of } A)^t$.
- A^{-1} exists iff $\det A \neq 0$. And $A^{-1} = \frac{1}{\det A} (\operatorname{adj} A)$.

Properties: For square matrices A & B of order n,

- $\det A = \det A^t$
- If A^{-1} exists, $\det(A^{-1}) = \frac{1}{\det A}$
- $\det(AB) = (\det A) \cdot (\det B)$
- Determinant of an upper (a lower) triangular matrix = Product of diagonal elements.
- If any two rows (columns) are linearly dependent, then determinant of the matrix is zero.
- $\det(\alpha A) = \alpha^n (\det A)$; $\alpha \in \mathbb{R}$.
- If B is obtained from A by interchanging two rows (columns), then $\det B = -\det A$.
- If B is obtained from A by multiplying one row (column) by constant k, then $\det B = k(\det A)$.
- If B is obtained from A by adding a constant multiple of one row (column) to another row (column), then $\det B = \det A$.

e.g.:

1) Find A^{-1} & B^{-1} if exist, where $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$.

2) Find $k \in \mathbb{R}$ so that $A = \begin{pmatrix} 1 & 2 & k \\ 3 & -1 & 1 \\ 5 & 3 & -5 \end{pmatrix}$ is Singular.

3) Find $\det(BA^{-1}B^T)$, where $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$ & $B = \begin{pmatrix} -3 & 6 \\ 3 & 2 \end{pmatrix}$.

4) Find $\det A$ & $\det B$, where $\det(A^2 B) = 16$ & $\det(A^t B) = 2$.

5) Find $\det(2A)$, where $\det A = 3$ & it is a square matrix of order 5.

6) Find $\det(3B)$, where $\det A = 3$, $A = \begin{pmatrix} 1 & x & 0 \\ 2 & y & 1 \\ 3 & z & -1 \end{pmatrix}$ &

$$B = \begin{pmatrix} 1 & x+1 & 0 \\ 2 & y+2 & 1 \\ 3 & z+3 & -1 \end{pmatrix}.$$

Cramer's Rule

Finding solution to a system of n linear equations with n unknowns using determinant.

Non-Homogeneous System

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = b_n$$

Homogeneous System

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = 0$$

OR

$$Ax = b \qquad | \qquad Ax = 0$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ & $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$.

- If $\det A \neq 0$, then the system has a unique solution, where $x_i = \dfrac{\det A(i)}{\det A}$, $i = 1,2,3,\dots,n$, where $A(i)$ is the matrix obtained from A by replacing i^{th} column of A by b.
- For the homogeneous system, if $\det A \neq 0$, then $x_i = 0$ for all $i = 1,2,3,\dots,n$. That is, it has only the zero solution called trivial solution.
  Thus, it has also non-zero (non-trivial) solution iff $\det A = 0$.

e.g.: Solve
$$x + 2y - 3z = 1$$
$$2x - y + z = 4$$
$$x + 3y = 5.$$

$x \simeq$
$y \simeq 1$
$z \simeq 1$

Elementary Row operations

- A matrix is said to be in Echelon form, if leading non-zero element of a row is to the left of leading non-zero element of the successive rows, and a zero row is at the bottom (if any).
- We can transform any matrix into its echelon form by using either of the following three elementary row operations. These are:
  - Interchanging any two rows
  - Multiplying a row by a non-zero constant
  - Adding a scalar multiple of one row to another row.
- If matrix B is obtained from matrix A by applying a finite number of elementary row operations on A, then we say B is row equivalent to A.

e.g.: Find echelon form of the following matrices.

(1) $A = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 2 & 4 \\ 4 & 0 & 0 \end{pmatrix}$ (2) $B = \begin{pmatrix} 3 & -10 & 5 \\ -1 & 12 & -2 \\ 1 & -5 & 2 \end{pmatrix}$ (3) $C = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

Gaussian Elimination Method.

- Let
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$
be a system of m linear equations with n unknowns (variables) $x_1, x_2, \ldots, x_n$.

or, in matrix representation $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \ \& \ b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

- To determine solution to this system of linear equations, reduce the augmented matrix $(A|b)$ into its Echelon form, and use back substitution.
- The system is said to be consistent if it has a common solution. Otherwise, inconsistent.

E.g: Solve the following system of linear equations, if consistent.

1)
$$\begin{aligned} x + y + z &= 7 \\ x + 2y + 3z &= 16 \\ x + 3y + 4z &= 22 \end{aligned}$$

3)
$$\begin{aligned} x - 3y - 8z &= -10 \\ 3x + y - 4z &= 0 \\ 2x + 5y + 6z &= 13. \end{aligned}$$

2)
$$\begin{aligned} 2x - 5y + 7z &= 6 \\ x - 3y + 4z &= 3 \\ 3x - 8y + 11z &= 11 \end{aligned}$$

Gaussian Method of inverting a matrix

For an invertible square matrix A of order n, reduce the augmented matrix $(A|I_n)$ to $(I_n|B)$ by using elementary row operations.

Then $B = A^{-1}$.

E.g: Find multiplicative inverse of $A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$.

Rank of a Matrix

- Rank of matrix A, denoted $R(A)$, is the order of the largest possible square submatrix, whose determinant is different from zero.
- If A is an $m \times n$ matrix, then $R(A) \le \min\{m, n\}$.
- If A is a non-singular square matrix of order n, then $R(A) = n$.
- Rank of a zero matrix is zero.
- If matrix B is row equivalent to matrix A, then $R(B) = R(A)$.

E.g.

1) Find rank for

(a) $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ (c) $C = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & -1 & 3 \end{pmatrix}$.

2) Find $k \in \mathbb{R}$ so that rank of the following is 2.

(a) $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 4 & 2 \\ 3 & 5 & k \end{pmatrix}$ (b) $B = \begin{pmatrix} 2 & 1 & 1 & 2 \\ 3 & 0 & -1 & 4 \\ 7 & 2 & k & 8 \end{pmatrix}$.

3) Find $p, q \in \mathbb{R}$ so that rank of $A = \begin{pmatrix} 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \\ 6 & -2 & p & q \end{pmatrix}$ is

(a) 3 (b) 2.

- Consider a system of m linear equations with n unknowns (variables) given by $Ax = b$, where $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, & $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

(i) If $R(A|b) \neq R(A)$, the system is inconsistent & hence has no solution.

(ii) If $R(A|b) = R(A) = n$ (no. of unknowns), the system is consistent & has a unique solution.

(iii) If $R(A|b) = R(A) < n$, the system is consistent & has infinitely many solutions.

- For homogeneous system $Ax = 0$, it has non-trivial solutions iff $R(A) < n$.

E.g:

1) Find $\lambda, \beta \in \mathbb{R}$ so that the following have unique, infinitely many & no solutions.

(a)
$$\begin{aligned} x - y + 2z &= -3 \\ 2x + y + 3z &= 1 \\ 4x - y + 7z &= \lambda \end{aligned}$$

(b)
$$\begin{aligned} x - 7y + 8z &= 17 \\ 3x + y + \lambda z &= \beta \\ 2x - 3y + 5z &= 12 \end{aligned}$$

(c)
$$\begin{aligned} x - 2y + \lambda z &= -1 \\ \lambda x - 2y + z &= -1 \\ x - 2\lambda y + z &= 2 \end{aligned}$$

2) Find $\lambda \in \mathbb{R}$ so that the following have non-trivial solution.

(a)
$$\begin{aligned} x + (\lambda - 3)y &= 0 \\ (\lambda - 3)x + y &= 0 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 + x_2 - \lambda x_3 &= 0 \\ 4x_1 - 2x_2 - 3x_3 &= 0 \\ 2\lambda x_1 + 4x_2 + \lambda x_3 &= 0. \end{aligned}$$

CHAPTER-3
Limit & Continuity

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Limits

Let f be function of x, i.e. $f(x)$.

- $f(a) = L$ represents L is the value of f at a.
- $\lim_{x \to a} f(x) = L$ represents
  - L is the limit of f at a.
  - $f(x)$ approaches L as x approaches a.
  - $f(x)$ gets arbitrarly close to L for values of x sufficiently close to a.
  - L is the value of f near a (not at a).
- $\lim_{x \to a^+} f(x) = L$ represents
  - L is the right hand limit of f at a.
  - $f(x)$ approaches L as x approaches a from the right.
  - L is the value of f for values of x close to a from the right.
- $\lim_{x \to a^-} f(x) = L$ represents L is the left hand limit of f at a.
- $\lim_{x \to a^+} f(x) = \infty$ represents f gets arbitrarly large for values of x close to a from the right.
- $\lim_{x \to a^+} f(x) = -\infty$ represents f gets arbitrarly small for x's close to a from the right.
- $\lim_{x \to a^-} f(x) = \infty$
- $\lim_{x \to a^-} f(x) = -\infty$
- $\lim_{x \to \infty} f(x) = L$ represents f gets arbitrarly close to L for sufficiently large values of x.
- $\lim_{x \to -\infty} f(x) = L$ represents f approaches L for very small values of x.
- $\lim_{x \to \infty} f(x) = \infty$ represents f gets arbitrarly large for sufficiently large x.
- $\lim_{x \to -\infty} f(x) = \infty$ represents f gets arbitrarly large for sufficiently small x.
- $\lim_{x \to \infty} f(x) = -\infty$
- $\lim_{x \to -\infty} f(x) = -\infty$.

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Note:

(i) $\lim_{x\to a} f(x) = L$ iff for all $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

(ii) $\lim_{x\to a^+} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0$ such that if $0 < x-a < \delta$ then $|f(x) - L| < \varepsilon$.

(iii) $\lim_{x\to a^-} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta > 0$ such that if $-\delta < x-a < 0$ then $|f(x) - L| < \varepsilon$.

(iv) $\lim_{x\to a} f(x)$ exists iff both one-sided limits $\lim_{x\to a^+} f(x)$ & $\lim_{x\to a^-} f(x)$ exist, & are equal.

e.g: Show that

1) $\lim_{x\to 2} 2x = 4$

2) $\lim_{x\to -1} 4x+3 = -1$

3) $\lim_{x\to 0} x^2 = 0$

4) $\lim_{x\to 1} x^2+2x+1 = 4$

5) $\lim_{x\to 1} \frac{x+1}{2x+1} = \frac{2}{3}$

6) $\lim_{x\to a} |x| = |a|$

7) $\lim_{x\to 2} \sqrt{x} = \sqrt{2}$

8) $\lim_{x\to 1^+} \sqrt{x-1} = 0$

9) $\lim_{x\to 1^-} \sqrt{1-x} = 0$

10) $\lim_{x\to 1^+} \frac{x}{x-1} = \infty$

11) $\lim_{x\to 1^-} \frac{x}{x-1} = -\infty$

12) $\lim_{x\to -\infty} \frac{1}{x} = 0$

13) $\lim_{x\to \infty} \sqrt{\frac{x+1}{x-1}} = 1.$

Determining Limit

We can evaluate limit of a function at any number, ∞ or $-\infty$, by using either of the following techniques.

+ Factorization :- Factorize numerator & denominator, & cancel common factors.

e.g: Find $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

+ Rationalization :- In functions that involve radicals, rationalize & cancel common factors. e.g: Find $\lim_{x\to 1} \frac{1-\sqrt{x}}{x-1}$.

+ Substitution :- Technique of writing a limit to a simpler form.

If $\lim_{x\to a} h = L$ & $\lim_{h\to L} f(h) = M$, then $\lim_{x\to a} f(h) = M$.

e.g: Find $\lim_{x\to 0} \sin(2x + \frac{\pi}{2})$.

$\sin(\pi/2) = 1$

Note: $\lim_{x\to a} f(x) = \lim_{h\to 0} f(h+a)$.

+ Squeezing (Sandwich) theorem :- If $f(x) \le h(x) \le g(x)$ &

$\lim_{x\to a} f(x) = L = \lim_{x\to a} g(x)$, then $\lim_{x\to a} h(x) = L$.

e.g: Find $\lim_{x\to 0} x^2 \sin(1/x)$

- If $f(x) = \begin{cases} g(x), & \text{if } x \le a \\ h(x), & \text{if } x > a \end{cases}$, then

$\lim_{x\to a} f(x)$ exists iff both $\lim_{x\to a^+} f(x)$ & $\lim_{x\to a^-} f(x)$ exist & are equal.

e.g: Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x, & \text{if } x \le 1 \\ \sqrt{x}, & \text{if } x > 1 \end{cases}$

- $\lim_{x\to 0} \frac{\sin x}{x} = 1.$ $\quad \lim_{x\to\infty} \frac{\sin(1/x)}{1/x}$, $\lim_{x\to 0} (1+x)^{1/x} = e$

- $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e.$ $\quad \lim_{x\to\infty} (1+\frac{1}{x})^{x} = e$

e.g. Evaluate the following limits (if exist).

1) $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$ $= 3$

2) $\lim_{x\to 4^-} \frac{16-x^2}{|x-4|}$ -8

3) $\lim_{x\to -\infty} \frac{3-2|x|}{3x-4}$ $\to 2/3$

4) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$ $\to 1/2$

5) $\lim_{t\to 3} \frac{t\sqrt{t+1}-6}{t-3}$ $\to 11/4$

6) $\lim_{x\to 0} \frac{\tan x}{x}$ $- 1$

7) $\lim_{x\to 0} \frac{\sin(3x)}{x}$ $- 3$

8) $\lim_{x\to 0} \frac{\sin^2(x/3)}{x^2}$ $- \frac{1}{9}$

9) $\lim_{x\to 0} \frac{2-\sqrt{x+4}}{\sin(2x)}$ $- 1/8$

10) $\lim_{x\to 0} \frac{\cos x-1}{x}$ $- 0$

11) $\lim_{x\to \pi/2} \frac{1-\sin x}{x-\pi/2}$ $- 0$ $\quad y$ $\quad y\to 0$

12) $\lim_{x\to 2} (x-1)^{(\frac{1}{x-2})}$

13) $\lim_{x\to 1} x^{(\frac{1}{1-x})}$

14) $\lim_{x\to 0} \frac{|x|}{x^2+2x}$

15) $\lim_{x\to \infty} \frac{x}{|x|-1}$

16) $\lim_{x\to 1} \frac{x+1}{|x-1|}$

$\lim_{x\to\infty} (1+\frac{1}{x})^{x} = e$

$\lim_{x\to\infty} (1+\frac{4}{x})^{x} = e^4$

$\lim_{x\to 0} (1+4x)^{1/x} = e^4$

$f(0)$

17) $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} x^2-3x+1, & \text{if } x<0 \\ 2x^3-1, & \text{if } x \ge 0 \end{cases}$ doesn't exist

18) $\lim_{x\to 0} g(x)$, where $g(x) = \begin{cases} \frac{x}{|x|+x^2}, & \text{if } x \ne 0 \\ 0, & \text{if } x=0 \end{cases}$ doesn't exist

19) $\lim_{x\to 0} h(x)$, where $h(x) = \begin{cases} \frac{|x|^2-x^2}{x^3+x^2}, & \text{if } x\ne 0 \\ 0, & \text{if } x=0 \end{cases}$

20) $\lim_{x\to 0} f(x)$, where $x+\cos x \le f(x)\cdot(x^2+2) \le x+1$. — squeezing

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Infinite limits & Vertical asymptotes

If function f satisfies either $\lim_{x\to a^+} f(x) = \infty$, $\lim_{x\to a^+} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \infty$ or $\lim_{x\to a^-} f(x) = -\infty$, then the vertical line $x = a$ is called a vertical asymptote to the graph of f.

Limit at Infinity & Horizontal asymptotes

If function f satisfies either $\lim_{x\to \infty} f(x) = L$ or $\lim_{x\to -\infty} f(x) = L$, $L \in \mathbb{R}$, then the horizontal line $y = L$ is called a horizontal asymptote to the graph of f.

Eg: Find V.A & H.A to graph of the following functions (if any).

1) $\boxed{f(x) = \dfrac{2x}{1-x}}$

2) $g(x) = \sqrt{\dfrac{x+1}{x-1}}$ $\dfrac{x^2}{2x^2-1}$

3) $h(x) = \dfrac{x|x|}{2x^2-1}$ 2 H.A & 2 V.A $\frac{1}{2}, -\frac{1}{2}$ $x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$

4) $f(x) = \dfrac{x}{|x| + x^2}$

5) $g(x) = \sqrt{1+x} - \sqrt{x}$ no V.A $y=0$ – H.A

6) $h(x) = \dfrac{\sqrt{1+x^2}}{x}$ $y=1,-1$ – H.A V.A $x=0$

7) $f(x) = \dfrac{x}{\sqrt{4x^2+1} - 1}$ $x=0$, V.A $y = \frac{1}{2}, -\frac{1}{2}$ H.A

Continuity

- Function f is said to be continuous at $x = a$ iff $\lim_{x\to a} f(x) = f(a)$. Otherwise, discontinuous at $x = a$.
- Function f is said to be continuous at $x = a$
  - from the right iff $\lim_{x\to a^+} f(x) = f(a)$
  - from the left iff $\lim_{x\to a^-} f(x) = f(a)$.
- A continuous function is a function that is continuous at each point of its domain.
- Function f continuous on (a, b), if it is continuous at each point of (a, b).
- Function f continuous on $[a, b]$, if it is
  - continuous on (a, b)
  - continuous at a from the right, &
  - continuous at b from the left.

E.g:

1) Determine Continuity of the following functions at indicated numbers.

(a) $f(x) = \begin{cases} \frac{x^2+x-2}{x+2}, & \text{if } x \neq -2 \\ -3, & \text{if } x=-2 \end{cases}$ at $x=-2$.

(b) $g(x) = \begin{cases} \frac{x^2}{2}-2, & \text{if } x<2 \\ 0, & \text{if } x=2 \\ 2-\frac{4}{x}, & \text{if } x>2 \end{cases}$ at $x=2$

(c) $h(x) = \begin{cases} \frac{2|x|+x^2}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x=0 \end{cases}$ at $x=0$

right, 2 left, -2

(d) $f(x) = \sqrt{|x|-x}$ at $x=0$

2) Find values of Constants so that the following functions are continuous functions.

(a) $f(x) = \begin{cases} \frac{\tan(3x)}{x}, & \text{if } x \neq 0 \\ a+2, & \text{if } x=0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{4\sin(ax)}{x}, & \text{if } x<0 \\ x^2-a^2, & \text{if } x \geq 0 \end{cases}$

(c) $h(x) = \begin{cases} x^2+a, & \text{if } x \geq 0 \\ -x^2-a, & \text{if } x<0 \end{cases}$

(d) $f(x) = \begin{cases} \frac{4-\sqrt{4x}}{x-4}, & \text{if } x \neq 4 \\ 6a, & \text{if } x=4 \end{cases}$

(e) $g(x) = \begin{cases} ax^2+bx, & \text{if } x<1 \\ 3b, & \text{if } x=1 \\ 2x+b, & \text{if } x>1. \end{cases}$

$\lim_{x\to 1^-} ax^2+bx = \lim_{x\to 1^+} 2x+b = g(1)$

$a+b = 2+b$

$a+\cancel{b}-\cancel{b} = 2$

$a=2$

$2x+b=3b$

$2+b=3b$

$3b=b+2$

$3b-b=2$

$2b=2$

$b=1$

Intermediate Value Theorem (IVT):

If function f is continuous on $[a,b]$, and P is any number between $f(a)$ & $f(b)$, then there is a number c between a & b such that $f(c)=P$.

E.g: Show that equation

1) $x^3+x-1=0$ has at least one root on the interval $[-1,1]$.

2) $2\sin x = \sqrt{2}\cos x$ has at least one solution on the interval $[-\pi, 0]$.

3) $\cos x = x$ has at least one solution.

4) $x^3-15x+1=0$ has at least three solutions on the interval $[-4,4]$.

$(x^2-x)^x$ $\left(\frac{x^2+2x+1}{x^2+3x+1}\right)^x$

CHAPTER-4
Differentiation

Defn: Let a be a number in the domain of function $f(x)$. If $\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$ exists, we call this limit the derivative of f at a, denoted $f'(a)$.

That is, $f'(a)=\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}$. And, we say f is differentiable at a.

- For any x in the domain of f, $f'(x)=\lim\limits_{t\to x}\dfrac{f(t)-f(x)}{t-x}$ if the limit exists.
- Function f is said to be:
  - differentiable on (a,b), if it is differentiable at each point of (a,b).
  - differentiable on $[a,b]$, if it is differentiable on (a,b), differentiable at a from the right & differentiable at b from the left.
  - a differentiable function, if it is differentiable at each point of its domain.
- The 2^{nd} & 3^{rd} derivatives of f, $f''(x)$ & $f'''(x)$ respectively, are defined as

$$f''(x)=\lim_{t\to x}\frac{f'(t)-f'(x)}{t-x} \quad \& \quad f'''(x)=\lim_{t\to x}\frac{f''(t)-f''(x)}{t-x}$$, if the limits exist.

- If we let $y=f(x)$, then

  $f'(x)$ can also be denoted as $\dfrac{d\,f(x)}{dx}=\dfrac{dy}{dx}$.

  $f''(x)$ can also be denoted as $\dfrac{d}{dx}\left(\dfrac{d\,f(x)}{dx}\right)=\dfrac{d^2f(x)}{dx^2}=\dfrac{d^2y}{dx^2}$.

  $f'''(x)$ can also be denoted as $\dfrac{d}{dx}\left(\dfrac{d^2f(x)}{dx^2}\right)=\dfrac{d^3f(x)}{dx^3}=\dfrac{d^3y}{dx^3}$.

E.g: Find derivative of the following functions.

1) $f(x)=x^2+3x$
2) $g(x)=\dfrac{1}{x}$
3) $h(x)=\dfrac{x}{x+1}$
4) $f(x)=\sqrt{x}$
5) $g(x)=|x-1|$
6) $h(x)=x|x|$
7) $f(x)=\sin x$
8) $g(x)=\cos x$. $-\sin$
9) $h(x)=\log_2^x$.
10) $f(x)=2^x$

Note:

(i) If f is a constant function, then $f'(x)=0$ for all x.

(ii) If $f(x)=x^n$ for any n, then $f'(x)=nx^{n-1}$

(iii) If function f is differentiable at a, then it is continuous at a. But the converse need not be true.

E.g.: Find $a, b \in \mathbb{R}$ so that the following are differentiable functions.

1) $f(x) = \begin{cases} x^2, & \text{if } x \le 1 \\ ax+b, & \text{if } x > 1 \end{cases}$

2) $g(x) = \begin{cases} ax-6, & \text{if } x > 1 \\ bx^2, & \text{if } x \le 1 \end{cases}$

3) $h(x) = \begin{cases} x^2+3x+a, & \text{if } x \le 1 \\ bx+2, & \text{if } x > 1 \end{cases}$

4) $f(x) = \begin{cases} \frac{1}{|x|}, & \text{if } |x| \ge 1 \\ ax^2+b, & \text{if } |x| < 1. \end{cases}$

Derivative of Combination of Functions

- **Addition Rule:**

$(f+g)'(x) = f'(x) + g'(x)$ for all x at which both f & g are differentiable.

- **Constant multiple Rule:**

$(kf)'(x) = k f'(x)$ for all x at which f is differentiable & $k \in \mathbb{R}$.

- **Product Rule:**

$(f \cdot g)'(x) = f'(x)\,g(x) + f(x)\,g'(x)$ for all x at which both f & g are differentiable.

- **Quotient Rule:**

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

for all x at which both f & g are differentiable, and $g(x) \neq 0$.

E.g.: Find derivative of the following functions.

1) $f(x) = x^2 + \sin x + \cos x$

2) $g(x) = 4\sqrt{x}$

3) $h(x) = x^2 \sin x \cos x$

4) $f(x) = \dfrac{x^2+1}{x - \cos x}$

5) $g(x) = \tan x$

Derivative of Composition of Functions

- **Chain Rule:**

If f is differentiable at x & g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x &

$$(g \circ f)'(x) = (g(f(x)))' = g'(f(x)) \cdot f'(x).$$

E.g.:

1) Find derivative of the following functions.

(a) $f(x) = (2x^2+3x)^5$

(b) $g(x) = \sqrt{1+x^4}$

(c) $h(x) = e^{-x} \ln(1-x^2)$

(d) $\cancel{f(x) = \ln(\sqrt{1+x^2})}$

(e) $g(x) = \cos^3(4x)$

(f) $h(x) = e^{\sqrt{1+x^2}}$

(g) $f(x) = 5^{\ln(\sin x)}$

(h) $\cancel{g_1(x) = x^x}$

(i) $\cancel{h(x) = (\sin x)^{\sqrt{x}}}$

2) Find the n^{th} order derivative of $f(x) = \ln(1-x)$.

3) Let $h(x) = g(f(x))$ such that $f(2) = 3$, $f'(2) = 2$ & $g'(3) = 4$. Find $h'(2)$.

4) Let $f(a) = 1$ & $f'(a) = 4$. Find the value of the derivative of $(f(x))^5$ wrt x at a.

5) Let $h(x) = \sqrt{(f(x))^2 + (g(x))^2}$ with $f(2) = 4$, $f'(2) = \frac{1}{2}$, $g(2) = 2$ & $g'(2) = 3$. Find $h'(2)$.

6) Let $y = \left(\frac{1}{x}\right)^x$. Find $\frac{d^2y}{dx^2}$ at $x = 1$.

7) Let $y = \frac{1}{x}\sin x$. Show that $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$.

8) Suppose y is a differentiable function of x. Express the derivative of the following functions wrt x, interms of x, y & $\frac{dy}{dx}$.

(a) y^3 (b) $\frac{1}{y}$ (c) $\sin y$ (d) $\sin^3 y$ (e) $2x^3y^4$.

Implicit Differentiation

For a differentiable function y of x defined implicitly interms of x, we determine $\frac{dy}{dx}$ using implicit differentiation. That is, differentiating both sides of the equation wrt x & solving $\frac{dy}{dx}$.

E.g:

1) Find $\frac{dy}{dx}$ at indicated points, where y is a differentiable function of x.

(a) $x^2y + 3y - 3 = y^2$ at $(-1, 1)$

(b) $\sin x = \cos y$ at $(\pi/6, \pi/3)$

(c) $y^3 + \sin(xy^2) = \frac{3}{2}$ at $(\pi/6, 1)$.

2) Let $x^2 + y^2 = 1$, where y is a differentiable function of x. Write $\frac{d^2y}{dx^2}$, interms of y only.

3) Suppose x & y are both differentiable functions of t, and $y = \sin(xy^2)$. Find $\frac{dy}{dt}$, interms of x, y & $\frac{dx}{dt}$.

Geometric Interpretation of Derivative

If function f is differentiable at $x=a$, then $f'(a)$ represents the slope of the tangent line to the graph of f at the point $(a, f(a))$.

Thus, the slope of the normal line to the graph of f at the point $(a, f(a))$ is $-\frac{1}{f'(a)}$, Provided $f'(a) \neq 0$.

- If f is continuous at a & $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \infty$ (or $-\infty$), then the tangent line is the vertical line $x = a$.

e.g:

1) Find equations of tangent & normal lines, if any, to graph of the following functions at indicated points.

(a) $f(x) = \sqrt{x}$ at $(1,1)$

(b) $g(x) = \cos x$ at $(0,1)$

(c) $h(x) = \sqrt[3]{x+1}$ at $(-1,0)$

(d) $f(x) = \begin{cases} \sqrt{x}, & \text{if } 0 \le x \le 1 \\ x, & \text{if } x > 1 \end{cases}$ at $(1,1)$.

2) Find equations of tangent & normal lines, if any, to graph of the following curves at indicated points.

(a) $x^3 = y^2 + x^2 \sin y + 8$ at $(2,0)$

(b) $xy = \ln(x+y)$ at $(0,1)$

(c) $xe^y + y^2 x = \ln(e^x + y)$ at $(1,0)$.

3) Let $f(x) = 2x^3 - 9x^2 + 12x + 1$. Find point(s) on graph of f at which the normal line is vertical.

Arithmetic Interpretation of Derivative

$\frac{f(x)-f(a)}{x-a}$ represents the rate of change of f on the interval between a & x.

But, $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ represents the instantaneous rate of change of f wrt x at a.

For example, if V represents volume of an object which varies wrt time t, then $\frac{dV}{dt}$ represents an instant rate of change of V wrt t.

Thus, $\frac{dV}{dt} > 0$ represents V increases wrt time t, and

$\frac{dV}{dt} < 0$ represents V decreases wrt t.

e.g:

1) Height h of an object dropping from 144 meters wrt time t (in seconds) is given by $h = 144 - 16t^2$, until the object hits the ground.

Find the velocity of the object when it hits the ground.

2) A car is moving in a straight line. At time t (in seconds) its position S (in kms) is $S(t) = \frac{1}{100} t^3$ where $t \in [0,5]$.

Find time t, where the instantaneous velocity of the car is equal to its average velocity.

3) Area of a circle is increasing at a rate of 4π meters square per minute. How fast is its radius changing when radius is 2 meters.

4) Radius of a spherical balloon is increasing at a rate of $\frac{1}{10\pi}$ inches per minute. How fast is its volume changing when volume is 400π cubic inches.

5) One end of a 13 meter ladder is on the ground, & the other end rests on a vertical wall. If the bottom end is drawn away from the wall at 3 meters per second, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 5 meters far from the wall.

6) An airplane flying at an altitude of 5 miles is passing directly over a radar antenna. When the plane is 13 miles away from the radar, it is detected that the distance is increasing at a rate of 24 miles per hour. Find the actual speed of the plane.

Inverse Functions

- An inverse function of a given function is a function that reverses the correspondence of the given function.

If function f has an inverse with domain X & range Y, then its inverse, denoted f^{-1} is a function with domain Y & range X, and

$$f(x) = y \quad \text{iff} \quad x = f^{-1}(y) \quad \text{for every } x \in D_f \text{ \& every } y \in D_{f^{-1}}$$

Function f is also an inverse of f^{-1}.

- A function has an inverse iff it is one-to-one.
  (Geometrically, every horizontal line intersects G_f at most once).

For example:
$f(x) = x^3$ & $g(x) = \frac{1}{x}$ have inverse function on their domain, and
$h(x) = x^2$ & $k(x) = \sin x$ have no inverse on their domain.

- $G_{f^{-1}}$ is a reflection of G_f about the line $y = x$.

- If function f has an inverse, and
  (i) f is increasing, then f^{-1} is also increasing.
  (ii) f is decreasing, then f^{-1} is also decreasing.
  (iii) f is continuous, then f^{-1} is also continuous.
  (iv) f is differentiable with $f' \neq 0$, then f^{-1} is also differentiable.

Inverse Function Theorem:

Let function f has an inverse & be differentiable.
If $f(x) = y$, then $(f^{-1})'(y) = \frac{1}{f'(x)}$ provided $f'(x) \neq 0$.

$$\left(\text{i.e. if } f(a) = c\text{, then } (f^{-1})'(c) = \frac{1}{f'(a)}\right).$$

$$\text{And, } \frac{dx}{dy} = \frac{1}{dy/dx}.$$

E.g.

1) Show that the following functions have inverse on indicated intervals.
(a) $f(x) = 6x - 3$ on $(-\infty, \infty)$
(b) $g(x) = 1 - 4x^3$ on $(-\infty, \infty)$
(c) $h(x) = x^2 + 5x$ on $[-\frac{5}{2}, \infty)$
(d) $f(x) = \ln x$ on $(0, \infty)$.

2) Find inverse function for the following.
(a) $f(x) = 3 + x^3$
(b) $g(x) = \frac{x+1}{x-1}$
(c) $h(x) = \ln(2 + \ln x)$
(d) $f(x) = \sqrt{3 - e^{2x}}$.

3) Find $(f^{-1})'$ & $(g^{-1})'$, where $f(x) = 3x - 1$ & $g(x) = \frac{1}{x}$.

4) Let $f(x) = 5x^4 + x^6 - 2x^5 + x^2 - 7$ & $f(-1) = -8$. Find $(f^{-1})'(-8)$.

5) Find $(g^{-1})'(2)$, where $g(x) = x + \sqrt{x}$.

6) Find dx/dy for (a) $y = x^2 + 5x$

(b) $y = x \sin x$

Inverse Trigonometric Functions

- Inverse sine function, denoted arcsin ($\sin^{-1}$), is a function with domain $[-1, 1]$ & range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ defined as

$$\sin^{-1}(\sin x) = x \quad \text{for every } x \in [-\pi/2, \pi/2].$$

i.e. $\sin(\sin^{-1}x) = x$ for every $x \in [-1, 1]$, OR
$y = \sin^{-1}x$ iff $x = \sin y$ for every $x \in [-1, 1]$ & every $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

<u>Note</u>: $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ for every $x \in (-1, 1)$.

- Inverse Cosine function, denoted arccos (or $\cos^{-1}$), is a function with domain $[-1, 1]$ & range $[0, \pi]$ defined as

$$\cos^{-1}(\cos x) = x \quad \text{for every } x \in [0, \pi].$$

i.e. $\cos(\cos^{-1}x) = x$ for every $x \in [-1, 1]$, OR
$y = \cos^{-1}x$ iff $x = \cos y$ for every $x \in [-1, 1]$ & $y \in [0, \pi]$

<u>Note</u>: $\frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$ for every $x \in (-1, 1)$.

- The other inverse trigonometric functions are:

(a) $y = \tan^{-1}x$ iff $\tan y = x$ for every $x \in (-\infty, \infty)$ & $y \in (-\pi/2, \pi/2)$.

- $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ for every $x \in (-\infty, \infty)$

(b) $y = \cot^{-1}x$ iff $\cot y = x$ for every $x \in (-\infty, \infty)$ & $y \in (0, \pi)$.

- $\frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$ for every $x \in (-\infty, \infty)$.

(c) $y = \sec^{-1}x$ iff $\sec y = x$ for every x with $|x| \geq 1$ & $y \in \{[0, \pi/2) \cup [\pi, 3\pi/2)\}$.

- $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$ for every x with $|x| > 1$.

(d) $y = \csc^{-1}x$ iff $\csc y = x$ for every x with $|x| \geq 1$ & $y \in \{(0, \pi/2] \cup (\pi, 3\pi/2]\}$.

- $\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}$ for every x with $|x| > 1$.

Ex:

1) Find

(a) $\sin^{-1} 1$

(b) $\sin^{-1}(-1)$

(c) $\sin^{-1}(\sin(5\pi/6))$

(d) $\cos(\sin^{-1}(1/3))$

(e) $\sec(\sin^{-1}(-\frac{3}{4}))$

(f) $\cos^{-1}(\frac{1}{2})$

(g) $\cos^{-1}(\cos(5\pi/6))$

(h) $\sin(\cos^{-1}(-1/2))$

(i) $\tan^{-1}(\sqrt{3})$

(j) $\csc(\tan^{-1}(-\frac{1}{3}))$

(k) $\cos(\cos^{-1}(\frac{5}{13})+\sin^{-1}(\frac{3}{5}))$

(l) $\sin(\cos^{-1}(\frac{1}{2})+\tan^{-1}(\frac{3}{4}))$

2) Find derivative of the following.

(a) $\sin^{-1}(x^2-1)$

(b) $\tan^{-1}(\sec^2 x)$

(c) $\sqrt{\tan^{-1}(\sqrt{x})}$

(d) $\frac{-1}{\sin^{-1}x}$

(e) $x\tan^{-1}(\sqrt{x})$.

3) Show the following identities hold.

(a) $\sin^{-1}(-x) = -\sin^{-1}x$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x$

(c) $\tan^{-1}(-x) = -\tan^{-1}x$

(d) $\cos(\sin^{-1}x) = \sqrt{1-x^2}$

(e) $\sin(\cos^{-1}x) = \cos(\sin^{-1}x)$

(f) $\sec(\tan^{-1}x) = \sqrt{1+x^2}$

4) If $|ab| \neq 1$, show that

(a) $\tan^{-1}a + \tan^{-1}b = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$.

(b) $\tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$

Hyperbolic Functions

- The hyperbolic sine, denoted Sinh & the hyperbolic cosine, denoted Cosh, are defined as

$\sinh x = \frac{e^x - e^{-x}}{2}$, with domain $= \mathbb{R}$ & range $= \mathbb{R}$, and

$\cosh x = \frac{e^x + e^{-x}}{2}$, with domain $= \mathbb{R}$ & range $= [1, \infty)$.

- The other hyperbolic functions are:

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$, with domain $= \mathbb{R}$ & range $= (-1, 1)$.

$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$, with domain $= \mathbb{R}\setminus\{0\}$ & range $= \{(-\infty,-1)\cup(1,\infty)\}$.

$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$, with domain $= \mathbb{R}$ & range $(0, 1]$.

$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$, with domain $= \mathbb{R}\setminus\{0\}$ & range $= \{(-\infty, 0)\cup(0,\infty)\}$.

• Hyperbolic Identities:

(a) $\cosh^2 x - \sinh^2 x = 1.$

(b) $\operatorname{sech}^2 x + \tanh^2 x = 1.$

(c) $\coth^2 x - \operatorname{csch}^2 x = 1$

(d) $\sinh(-x) = -\sinh x$

(e) $\cosh(-x) = \cosh x$

(f) $\sinh(x \pm y) = \sinh x \cosh y \pm \sinh y \cosh x$

(g) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

(h) $\tanh(x \pm y) = \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

(i) $\sinh(2x) = 2\sinh x \cosh x$

(j) $\cosh(2x) = \cosh^2 x + \sinh^2 x.$

Note:

(i) $\frac{d}{dx}(\sinh x) = \cosh x$

(ii) $\frac{d}{dx}(\cosh x) = \sinh x$

(iii) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

(iv) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

(v) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

(vi) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x.$

e.g.:

1) Find derivative of the following functions.

(a) $\sinh(\sqrt{x^2+2x})$

(b) $\operatorname{sech}(e^{x^3+3x})$

(c) $e^{\sinh x^2}$

(d) $\dfrac{\tanh x}{1+x^2}$

2) Show that the following identities hold.

(a) $\cosh x + \sinh x = e^x$

(b) $\cosh x - \sinh x = e^{-x}$

(c) $\cosh(x/2) = \sqrt{\dfrac{(\cosh x)+1}{2}}$

(d) $\sinh(x/2) = \begin{cases} \sqrt{\dfrac{\cosh x - 1}{2}}, & \text{if } x \geq 0 \\ -\sqrt{\dfrac{\cosh x - 1}{2}}, & \text{if } x < 0 \end{cases}$

(e) $\cosh(\ln x) + \sinh(\ln x) = x$

(f) $(\cosh x + \sinh x)^n = \cosh(nx) + \sinh(nx)$

(g) $(\cosh x - \sinh x)^n = \cosh(nx) - \sinh(nx).$

3) Find dy/dx for

(a) $\ln(x+y) = \tan^{-1}(x/y)$

(b) $\cosh(x+y) = \sinh(xy).$

Inverse Hyperbolic Functions

- Inverse hyperbolic sine function, denoted arcsinh (or, $\sinh^{-1}$), is a function with domain $(-\infty, \infty)$ & range $(-\infty, \infty)$ defined as

$$\sinh^{-1}(\sinh x) = x \quad \text{for every } x \in (-\infty, \infty).$$

- $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ for every $x \in \mathbb{R}$.
- $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{1+x^2}}$ for every $x \in \mathbb{R}$.

- The other inverse hyperbolic functions are:

(a) $y = \cosh^{-1} x$ iff $\cosh y = x$, for every $x \in [1, \infty)$ & every $y \in [0, \infty)$.

- $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ for every $x \geq 1$.
- $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$ for every $x > 1$.

(b) $y = \tanh^{-1} x$ iff $\tanh y = x$, for every $x \in (-1, 1)$ & every $y \in (-\infty, \infty)$.

- $\tanh^{-1} x = \dfrac{1}{2} \ln\left(\dfrac{1+x}{1-x}\right)$ for $|x| < 1$
- $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2}$ for $|x| < 1$.

E.g.:

1) Find derivative of the following functions:

(a) $\cosh^{-1}(\sqrt{x})$

(b) $\tanh^{-1}(\tan x)$

(c) $\tan(\tan^{-1} x)$

(d) $\sinh(\sinh^{-1} x^2)$.

2) Show that $\tanh^{-1}\left(\frac{3}{5}\right) = \sinh^{-1}\left(\frac{3}{4}\right) = \cosh^{-1}\left(\frac{5}{4}\right)$.

Applications of Derivative

(I) L'Hopital's Rule: (determining limit of indeterminants using derivative).

Let f & g be differentiable functions of x, and c be any real number, ∞ or $-\infty$.

(i) / Indeterminate $\frac{0}{0}$ /:

If $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Provided this limit is finite or infinite.

(ii) / Indeterminate $\frac{\infty}{\infty}$ /:

If $\lim_{x\to c} f(x) = \pm\infty = \lim_{x\to c} g(x)$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Provided this limit is finite or infinite.

e.g: Evaluate the following limits.

1) $\lim_{x\to 1} \frac{e^x - e}{x^2-1}$

2) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

3) $\lim_{x\to 0^+} \frac{1-\cos\sqrt{x}}{\sin x}$

4) $\lim_{x\to \infty} \frac{1+x^2}{x\ln x}$

5) $\lim_{x\to 0} \frac{x-\sin x}{x^3\cos^2 x}$

6) $\lim_{x\to \infty} \frac{\ln(\ln x)}{x\ln x}$

7) $\lim_{x\to 1} \frac{1-x+\ln x}{1+\cos(\pi x)}$

8) $\lim_{x\to 0^+} \frac{\ln x}{\cot x}$

- Other indeterminates are of the form $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ & ∞^0. These indeterminates must be converted into indeterminates $\frac{0}{0}$ or $\frac{\infty}{\infty}$ to apply L'Hopital's Rule.

e.g. Evaluate the following limits.

1) $\lim_{x\to\infty} x(e^{1/x} - 1)$

2) $\lim_{x\to\infty} x\ln\left(\frac{x-1}{x+1}\right)$

3) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

4) $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right)$

5) $\lim_{x\to 0^+} x^x$

6) $\lim_{x\to 0^+} x^{\tan x}$

7) $\lim_{x\to 1^-} (1-x^2)^{\frac{1}{\ln(1-x)}}$

8) $\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}$

9) $\lim_{x\to 0^+} (1+\sin x)^{\cot x}$

10) $\lim_{x\to\infty} x^{\ln(1+\frac{1}{x})}$

11) $\lim_{x\to 0^+} (\csc x)^{\frac{1}{\ln x}}$

(II) Extreme values of a function on an interval

A function f is said to have a maximum value [or, a minimum value] on an interval I, if there is a number c in I [or, a number d in I] such that $f(c) \geq f(x)$ for all x in I [or, $f(d) \leq f(x)$ for all x in I].

$f(c)$ is called the maximum value & $f(d)$ is called the minimum value of f on I.

Max. & min. values are called extreme values of the function.

- Function f may or may not have extreme values on interval I, depending on f or on I. For example:

  $f(x) = \frac{1}{x}$ has no extreme values on interval $[-1, 1]$, &

  $g(x) = x$ has no extreme values on interval $(-1, 1)$.

- Extremum Value Theorem /maximum - minimum Theorem/:

  A continuous function on a closed interval have max. & min. values on that interval.

  For example: Functions $f(x) = x^2$, $g(x) = x^3$ & $h(x) = |x|$ have max. & min. values on $[-1, 3]$.

- A number x_0 in the domain of function f is called a critical number of f, if $f'(x_0) = 0$ or $f'(x_0)$ does not exist.

- Extreme values of a continuous function f on a closed interval $[a, b]$, occur at critical numbers on (a, b), or at the endpoints a or b.

  Thus, by comparing values of f at all critical nos on (a, b), & at endpoints a & b, the largest of the values computed is the maximum value & the smallest is the minimum value of f on $[a, b]$.

E.g: Find extreme values of the following functions on the indicated intervals.

1) $f(x) = x - x^3$ on $[0, 1]$
2) $g(x) = |x - 1|$ on $[0, 3]$
3) $h(x) = 2^x$ on $[0, 3]$
4) $f(x) = x^2 e^x$ on $[-1, 1]$
5) $g(x) = (x^2 - 1)^2$ on $[-2, 1]$.

(III) Monotonicity of a function

A function f is said to be:

- increasing on an interval I iff $f(x_1) \le f(x_2)$ whenever $x_1 < x_2$ for all x_1, x_2 in I.
- decreasing on an interval I iff $f(x_1) \ge f(x_2)$ whenever $x_1 < x_2$ for all x_1, x_2 in I.

Increasing & decreasing functions are called Monotonic Functions.

For example:

$f(x) = x^2$ is decreasing on $(-\infty, 0]$ & increasing on $[0, \infty)$, &
$g(x) = x^3$ is increasing on $(-\infty, \infty)$.

Note:

(i) If $f'(x) \ge 0$ for all x in I, then function f is increasing on I,

(ii) If $f'(x) \le 0$ for all x in I, then function f is decreasing on I.

Q.9: Determine interval of monotonicity for the following functions.

1) $f(x) = x^2 + x + 1$

2) $g(x) = 2x^3 - 3x^2 - 36x + 1$

3) $h(x) = \frac{1}{x}$

4) $f(x) = \sqrt{16 - x^2}$

5) $g(x) = x - \sin x$

6) $h(x) = \ln(4 - x^2)$

7) $f(x) = \sqrt{x - x^2}$

8) $g(x) = x|x|$.

(IV) Relative (local) extreme values of a function

A function f is said to have a relative max. value at x_0 [or, a relative min. value at x_0], if there is an interval I about x_0 such that $f(x_0) \geq f(x)$ for all x in I [or, $f(x_0) \leq f(x)$ for all x in I].

Relative max. & relative min. values are called Relative extreme values of the function.

- Relative extreme values of a function occur at critical numbers.

• /First Derivative Test/:

If function f is continuous on an interval containing x_0 & x_0 is a critical number of f, then

(i) if f' changes sign from +ve to -ve at x_0, then f has a relative maximum value at x_0.

(ii) if f' changes sign from -ve to +ve at x_0, then f has a relative minimum value at x_0.

(iii) if f' doesnot change sign at x_0, then f has neither relative max. nor relative min. at x_0.

• /Second Derivative Test/:

If function f is continuous on an interval containing x_0 & x_0 is a critical number of f with $f'(x_0) = 0$, then

(i) if $f''(x_0) < 0$, then f has a relative maximum value at x_0.

(ii) if $f''(x_0) > 0$, then f has a relative minimum value at x_0.

(iii) if $f''(x_0) = 0$, then no conclusion can be drawn.

4.1: Find relative extreme values of the following functions, if any.

1) $f(x) = x^3 + 3x^2 + 1$

2) $g(x) = x^3 + \frac{3}{x}$

3) $h(x) = 2x - 3\sqrt[3]{x^2}$

4) $f(x) = x|x|$

5) $g(x) = \sqrt{1 + |x|}$

6) $h(x) = \sin x$

7) $f(x) = x + \cos(2x)$.

(V.) Concavity & Inflection points

Graph of function f is said to be:

- concave upward at the point $(x_0, f(x_0))$, if there is an interval I containing x_0 such that graph of f on I is above the tangent line to the graph of f at $(x_0, f(x_0))$.
- Concave downward at $(x_0, f(x_0))$, if there is an interval I containing x_0 such that graph of f on I is below the tangent line to the graph of f at $(x_0, f(x_0))$.

Graph of f is concave upward on I, if it is concave upward at each point of interval I, & concave downward on I, if it is concave downward at each point of I.

For example:

Graph of $f(x) = x^2$ is concave upward on its domain, &

graph of $g(x) = x^3$ is concave downward on $(-\infty, 0)$ & concave upward on $(0, \infty)$.

→ An inflection point is a point on the graph of the function where the graph changes its concavity.

• /Concavity Test/:

Graph of function f is

(i) concave upward in I, if $f''(x) > 0$ for all x in I.

(ii) concave downward in I, if $f''(x) < 0$ for all x in I.

Eg: Determine interval of concavity, & find inflection points (if any).

1) $f(x) = x^4 - 6x^2$

2) $g(x) = 3x^5 - 5x^4 + 1$

3) $h(x) = \dfrac{2x-5}{x-1}$

4) $f(x) = x^2|x|$

5) $g(x) = \frac{1}{2}x^2 + \cos x$.

Curve sketching

General procedures to sketch graph of a function:

1) Find domain
2) Find intercepts (if any).
   (i) x-intercepts (ii) y-intercepts
3) Find asymptotes (if any).
   (i) Vertical asymptotes (ii) Horizontal asymptotes
4) Determine interval of monotonicity
5) Find relative extremes (if any).
6) Determine interval of concavity
7) Find inflection points (if any).

e.g.: Sketch graph of the following functions.

(a) $f(x) = \dfrac{x^2}{x^2-4}$ (b) $g(x) = \dfrac{2x-5}{x-1}$ (c) $h(x) = \dfrac{x^2}{(x-1)^2}$

Applications of Extreme values of a function on an interval

(A)

• **Rolle's Theorem**

If function f is continuous on $[a,b]$, differentiable on (a,b) & $f(a)=f(b)$, then there is a number $c \in (a,b)$ such that $f'(c)=0$.

• **Mean Value Theorem (MVT)**

If function f is continuous on $[a,b]$ & differentiable on (a,b), then there is a number $c \in (a,b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

e.g.:

1) Verify (a) Rolle's theorem for $f(x) = \sin x + \cos x$ on the interval $[0, 2\pi]$.
   (b) Rolle's theorem for $g(x) = (x-a)^3(x-b)^4$ on the interval $[a,b]$.
   (c) MVT for $h(x) = \frac{1}{3}x^3 + 2x$ on the interval $[0,3]$.

2) Show that
   (a) $|\sin x| \le |x|$ for all $x \in \mathbb{R}$
   (b) $|\sin x - \sin y| \le |x-y|$ for all $x, y \in \mathbb{R}$
   (c) $1 + x < e^x < 1 + xe^x$ for all $x > 0$.
   (d) $\dfrac{x}{x^2+1} < \tan^{-1}x < x$ for all $x > 0$

3) Let f be a polynomial function with $f'(x) \ne 0$ for all x.
   Show that equation $f(x)=0$ can have at most one root.

4) Show that between any two roots of $(x-1)\sin x = 0$, there is at least one root of equation $\tan x = 1-x$.

5) Let function f be continuous & differentiable on an interval I.
   If $|f'(x)| \le 1$ for all x in I, then show that
   $|f(x_1) - f(x_2)| \le |x_1 - x_2|$ for all x_1, x_2 in I.

(k) Maximization & Minimization problems

In some problems, the data may be given in two variables that are connected by some relation. In such problems, to find extreme values it is necessary to eliminate one variable with the help of the given relation.

eg:

1) Find two numbers whose sum is 8 & their product is maximum.

2) Find two non-negative numbers whose sum is 6 & sum of whose squares is minimum.

3) Find length & width of a rectangle with a perimeter of 20 meters & maximum area.

4) Two sides of a triangle are given. Find the angle between these two sides so that area of the triangle is maximum.

5) Sum of the lengths of a hypotenuse & another side of a right angled triangle is 9. Find the angle between these two sides so that area of the right angled triangle is maximum.

6) Surface area of a right circular cylinder is 54π. Find its radius & height so that it has a maximum volume.

7) 75 meters square of sheet metal is to be used in the construction of an open tank with square base. Find its dimensions so that the capacity is the greatest possible.

8) A land owner wishes to use 3 miles of fencing material to enclose an isosceles triangular region of as large an area as possible. What should be the lengths of the sides of the triangle.

9) Find the minimum possible area of a triangle determined by the x-axis, the y-axis & the line through the point (2, 1) in the first quadrant.

CHAPTER-5

Integration & its applications

- The set of all antiderivatives of a continuous function f is called the indefinite integral of f. That is,

  if $\frac{d}{dx}(F(x)) = f(x)$, then $\int f(x)\,dx = F(x) + c$, for an arbitrary constant c.

  integrand

- $\int_a^b f(x)\,dx$ is called the definite integral of f evaluated from a to b.

  That is, $\int_a^b f(x)\,dx = (F(x)+c)\Big|_{x=a}^{x=b} = F(b) - F(a)$.

Basic Integrations derived from Differentiation:

- $\int (f \pm g)(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
- $\int (kf)(x)\,dx = k\int f(x)\,dx$; $k \in \mathbb{R}$.
- $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$; $n \neq -1$.
- $\int x^{-1}\,dx = \ln|x| + c$; $x \neq 0$
- $\int a^x\,dx = \frac{a^x}{\ln a} + c$; $a > 0$ & $a \neq 1$.
- $\int \sin x\,dx = -\cos x + c$
- $\int \cos x\,dx = \sin x + c$
- $\int \sec x\,dx = \ln|\sec x + \tan x| + c$
- $\int \csc x\,dx = \ln|\csc x - \cot x| + c$
- $\int \sec^2 x\,dx = \tan x + c$
- $\int \csc^2 x\,dx = -\cot x + c$
- $\int \sec x \tan x\,dx = \sec x + c$
- $\int \csc x \cot x\,dx = -\csc x + c$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c$; for $|x|<1$.
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + c$
- $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c$; for $|x|>1$.
- $\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}x + c$
- $\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + c$; for $x>1$.
- $\int \frac{1}{1-x^2}\,dx = \tanh^{-1}x + c$; for $|x|<1$
- $\int \frac{1}{x\sqrt{1-x^2}}\,dx = -\operatorname{sech}^{-1}x + c$; for $0<x<1$.

Frequently used Trigonometric Identities:

- $\sin^2 x + \cos^2 x = 1$
- $1 + \tan^2 x = \sec^2 x$
- $1 + \cot^2 x = \csc^2 x$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$.

e.g:

1) Evaluate the following integrals.

(a) $\int 3(x+1)^2\, dx$

(b) $\int (4\cos x - 6x^2)dx$

(c) $\int_1^4 (t\sqrt{t} + \frac{3}{t^2})\, dt$

(d) $\int \tan^2 x\, dx$

(e) $\int \cot^2 x\, dx$.

2) Find equation of the curve $y = f(x)$:

(a) passing through $(0,1)$ & tangent line to the curve at any point (x,y) has slope x^2-2.

(b) passing through $(0,2)$ & $(1,3)$, satisfying $\frac{d^2y}{dx^2} = 6x-2$.

Techniques of Integration

(I) Integration using substitution

If
- f & g are functions with both $f\circ g$ & g' continuous, and
- F antiderivative of f,

then $\int f(g(x))\cdot g'(x)\, dx = F(g(x)) + C$.

Note: If variable $u = g(x)$, then $\frac{du}{dx} = g'(x)$, i.e. $du = g'(x)\,dx$.

Thus, $\int f(g(x))\cdot g'(x)\,dx = \int f(u)\,du = F(u)+C = F(g(x)) + C$.

And, $\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

e.g: Evaluate the following.

1) $\int 3(3x+4)^4\, dx$

2) $\int 2x\cos(x^2+2)\cdot dx$

3) $\int_0^1 x(x^2+1)^3 dx$

4) $\int \frac{1}{(1-x)^2}\, dx$

5) $\int \frac{1}{e^{2x-3}}\, dx$

6) $\int \sin(4x+2)\,dx$

7) $\int e^{\tan\theta}\cdot \sec^2\theta\, d\theta$

8) $\int \tan x\, dx$

9) $\int x\sqrt{x-2}\, dx$

10) $\int \frac{dx}{x+\sqrt{x}}$

11) $\int \sin x\cdot \cos^n x\, dx$, $n \neq -1$.

* Integration Using Trigonometric Substitutions

Integrals involving expressions of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ & $\sqrt{x^2-a^2}$ for constant a, can be simplified (that is, eliminate the radical) by making appropriate trigonometric substitutions based on

$$a^2 = a^2\sin^2\theta + a^2\cos^2\theta \quad \text{and} \quad a^2 + a^2\tan^2\theta = a^2\sec^2\theta.$$

For an integrand involving:

- $\sqrt{a^2-x^2}$; $a>0$, make the substitution $x = a\sin\theta$ where $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.
- $\sqrt{a^2+x^2}$, make the substitution $x = a\tan\theta$ where $\theta \in (-\pi/2, \pi/2)$
- $\sqrt{x^2-a^2}$, make the substitution $x = a\sec\theta$ where $\theta \in \{[0,\pi/2) \cup [\pi, \frac{3\pi}{2})\}$

θ is restricted to ensure $\frac{x}{a}$ is in the domain of the appropriate inverse trigonometric function.

Note: Integrands having expressions of the form $\sqrt{ax^2+bx+c}$ can be simplified by completing the square, & then applying trigonometric substitutions.

E.g: Evaluate the following.

1) $\int \sqrt{4-x^2}\,dx$

2) $\int \frac{\sqrt{9-x^2}}{x}\,dx$

3) $\int \frac{1}{\sqrt{9+x^2}}\,dx$

4) $\int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx$

5) $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$

6) $\int \frac{1}{\sqrt{x^2+2x+5}}\,dx$

7) $\int_1^2 \sqrt{3+2t-t^2}\,dt$

(II) Integration by parts

If f & g are differentiable functions, and f' & g' are continuous, then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx, \text{ OR}$$

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$$

Note: If variable $u = f(x)$ & variable $V = g(x)$, then $du = f'(x)\,dx$ & $dv = g'(x)\,dx$. Thus, integration by parts implies (or equivalent to), $\int u\,dv = uv - \int v\,du$.

E.g.: Evaluate the following.

1) $\int x\cos x\,dx$

2) $\int_0^1 x^2 e^x\,dx$

3) $\int x\ln x\,dx$

4) $\int \ln x\,dx$

5) $\int_0^{\pi} x^2\sin x\,dx$

6) $\int x^2 3^x\,dx$

7) $\int e^x\cos x\,dx$.

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* Reduction Formulas

Useful when the integrand involves higher powers of trigonometric functions.

(i) For $\int \sin^n x\,dx$, where n is a positive integer, let

$$f(x) = \sin^{(n-1)}(x) \quad \& \quad g'(x) = \sin x.$$

Hence $\int \sin^n x\,dx = -\sin^{(n-1)}(x)\cos x + (n-1)\int \sin^{(n-2)}(x)\cos^2 x\,dx$

$= -\sin^{(n-1)}(x)\cos x + (n-1)\int \sin^{(n-2)}(x)(1-\sin^2 x)\,dx$

Thus, we get $\int \sin^n x\,dx = -\frac{1}{n}\sin^{(n-1)}(x)\cos x + \frac{n-1}{n}\int \sin^{(n-2)}(x)\,dx$, for $n \geq 2$.

- Similarly, $\int \cos^n x\,dx = \frac{1}{n}\cos^{(n-1)}(x)\sin x + \frac{n-1}{n}\int \cos^{(n-2)}(x)\,dx$, for $n \geq 2$.

(ii) For $\int \tan^n x\,dx$, where n is a positive integer,

Since $\int \tan^n x\,dx = \int \tan^{(n-2)}(x)\tan^2 x\,dx = \int \tan^{(n-2)}(x)(\sec^2 x-1)\,dx$,

let $u = \tan x. \Rightarrow du = \sec^2 x\,dx$.

Thus, we get $\int \tan^n x\,dx = \frac{1}{n-1}\tan^{(n-1)}(x) - \int \tan^{(n-2)}(x)\,dx$, for $n \geq 2$.

- Similarly, $\int \cot^n x\,dx = -\frac{1}{n-1}\cot^{(n-1)}(x) - \int \cot^{(n-2)}(x)\,dx$, for $n \geq 2$.

(iii) For $\int \sec^n x\,dx$, where n is a positive integer, let

$$f(x) = \sec^{(n-2)}(x) \quad \& \quad g'(x) = \sec^2 x.$$

Hence $\int \sec^n x\,dx = \sec^{(n-2)}(x)\tan x - (n-2)\int \sec^{(n-2)}(x)\tan^2 x\,dx$

$= \sec^{(n-2)}(x)\tan x - (n-2)\int \sec^{(n-2)}(x)(\sec^2 x-1)\,dx$

Thus, we get $\int \sec^n x\,dx = \frac{1}{n-1}\sec^{(n-2)}(x)\tan x + \frac{n-2}{n-1}\int \sec^{(n-2)}(x)\,dx$, for $n \geq 2$.

- Similarly, $\int \csc^n x\,dx = -\frac{1}{n-1}\csc^{(n-2)}(x)\cot x + \frac{n-2}{n-1}\int \csc^{(n-2)}(x)\,dx$, for $n \geq 2$.

Note: Since negative integer powers of sine, cosine, tangent, cotangent, secant, cosecant are respectively positive integer powers of csc, sec, cot, tan, cos, sin, both positive & negative integer powers of these six functions can be integrated.

* Trigonometric Integrals

Useful when the integrand involves product of trigonometric functions.

(i) Consider $\int \sin^n x \cos^m x \, dx$, where n, m are positive integers.

(a) If n-odd & m-even, make the substitution $u = \cos x$.

(b) If n-even & m-odd, make the substitution $u = \sin x$.

(c) If n-odd & m-odd, make the substitution $u = \begin{cases} \cos x, & \text{if } n < m \\ \sin x, & \text{if } n > m. \end{cases}$

(d) If n-even & m-even, use $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ & $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

(ii) Consider $\int \tan^n x \sec^m x \, dx$, where n, m are positive integers.

(a) If n-odd/even/ & m-even, make the substitution $u = \tan x$

(b) If n-odd & m-odd, make the substitution $u = \sec x$.

(c) If n-even & m-odd, use integration by parts with $f'(x) = \sec x \tan x$.

(iii) (a) $\int \sin(ax)\sin(bx)\,dx = \frac{1}{2}\int (\cos(a-b)x - \cos(a+b)x)\,dx$.

(b) $\int \sin(ax)\cos(bx)\,dx = \frac{1}{2}\int (\sin(a+b)x + \sin(a-b)x)\,dx$.

(c) $\int \cos(ax)\cos(bx)\,dx = \frac{1}{2}\int (\cos(a+b)x + \cos(a-b)x)\,dx$.

E.g: Evaluate the following.

1) $\int \sin^3 x \cos^2 x \, dx$

2) $\int \sin^2 x \cos^5 x \, dx$

3) $\int \sin^3 x \cos^5 x \, dx$

4) $\int \sin^2 x \cos^2 x \, dx$

5) $\int \frac{\cos^3 x}{\sqrt{\sin x}} \, dx$

6) $\int \tan x \sec^4 x \, dx$

7) $\int \tan^5 x \sec x \, dx$

8) $\int \tan^2 x \sec x \, dx$

9) $\int \csc^4 x \cot^2 x \, dx$

10) $\int \sin(3x)\cdot\cos(7x)\,dx$

(III) Integration Using Partial Fractions

To determine partial-fraction decomposition of a rational function $\frac{P(x)}{Q(x)}$ with degree of $P(x) <$ degree of $Q(x)$:

(a) Factorize $Q(x)$ as a product of linear & irreducible quadratic factors.

(b) Each linear factor of $Q(x)$ of the form $(x+a)^n$ contributes a sum $\frac{A_1}{x+a} + \frac{A_2}{(x+a)^2} + \cdots + \frac{A_n}{(x+a)^n}$ to the decomposition, where $A_1, A_2, \ldots, A_n$ are constants.

(c) Each irreducible quadratic factor of $Q(x)$ of the form $(x^2+bx+c)^n$ contributes a sum $\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(x^2+bx+c)^n}$ to the decomposition, where $B_1, \ldots, B_n$ & $C_1, \ldots, C_n$ are constant.

(d) The total partial-fraction decomposition of $\frac{P(x)}{Q(x)}$ is the sum of all partial fractions contributed by (b) & (c).

E.g: Evaluate the following.

1) $\int \frac{2}{1-x^2}\,dx$

2) $\int \frac{x}{x^2+5x+6}\,dx$

3) $\int \frac{x^2+3x+3}{x(x+2)^2}\,dx$

4) $\int \frac{1}{x^3-x^2+x-1}\,dx$

5) $\int \frac{1}{x(1+x^2)^2}\,dx$

6) $\int \frac{2x^4-x^3-5x^2-3x+10}{x^3-x^2-4x+4}\,dx$.

Definite Integral

Let function $f(x)$ be continuous on the interval $[a,b]$. Let function $f(x)$ be continuous on the interval $[a,b]$.
Subdividing interval $[a,b]$ into n equal parts, each of width $\Delta x = \frac{b-a}{n}$, & choosing arbitrary number x_k from each k^{th} subinterval (for $k=1,2,\dots,n$), the Sum $[f(x_1)+f(x_2)+\dots+f(x_n)]\Delta x$ is called a Riemann Sum of f on $[a,b]$.

Thus, the definite integral of f on the interval $[a,b]$, denoted by $\int_a^b f(x)dx$, is the limit of the Riemann Sum as $n \to \infty$, that is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x.$$

Note:
(i) If function f is continuous on $[a,b]$, then $\int_a^b f(x)dx$ exists.
(ii) If function f is continuous on $[a,b]$ & $f(x)>0$, $\forall x \in (a,b)$, the definite integral $\int_a^b f(x)dx$ represents area of the region bounded by graph of f, vertical lines $x=a$ & $x=b$, and the x-axis.

The Fundamental Theorem of Calculus:

If (i) function f is continuous on the interval $[a,b]$, and
(ii) function F is an antiderivative (indefinite integral) of f,
(i.e. $F'(x) = f(x)$ for all $x \in [a,b]$)

then $\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b$.

Note: Let functions f & g be continuous on the interval $[a,b]$. Then

1) $\int_a^b (kf)(x)\,dx = k\int_a^b f(x)dx$, for a constant k.

2) $\int_a^b (f \pm g)(x)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

3) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4) $\int_a^a f(x)\,dx = 0$

5) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, for any $c \in [a,b]$.

E.g: Evaluate the derivative of the following wrt x.

a) $\int_1^x \sin u\,du$ b) $\int_0^{\sin x} u^3 du$ c) $\int_x^0 u^2\,du$ d) $\int_{x^2}^{4x} \cos u\,du$.

Improper Integrals

Definite integral $\int_a^b f(x)dx$ is called an improper integral, if either of the following holds:

(i) at least one of the limits of integration is infinite (i.e. ∞ or $-\infty$), or

(ii) f has one or more points of infinite discontinuity on $[a,b]$.

e.g: $\int_1^{\infty} 5x\,dx$, $\int_{-\infty}^{2} x^2 dx$, $\int_{-\infty}^{\infty} \frac{1}{x^2+1} dx$, $\int_0^5 \frac{1}{x} dx$, $\int_0^1 \frac{1}{(x-1)^2} dx$,

$\int_0^e \ln x\, dx$, $\int_0^4 \frac{1}{x-3} dx$, ... are improper integrals.

Note:

(i) If function f is continuous on $[a,\infty)$, then $\int_a^{\infty} f(x)dx = \lim_{M\to\infty} \int_a^M f(x)dx$.

Similarly, if f is continuous on $(-\infty, a]$, then $\int_{-\infty}^{a} f(x)dx = \lim_{M\to-\infty} \int_M^a f(x)dx$.

- If f is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{a} f(x)dx + \int_a^{\infty} f(x)dx, \text{ for any } a\in\mathbb{R}.$$

(ii) If function f is continuous on $(a,b]$ & has infinite discontinuity at a (i.e. $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$), then $\int_a^b f(x)dx = \lim_{M\to a^+} \int_M^b f(x)dx$.

Similarly, if f is continuous on $[a,b)$ & has infinite discontinuity at b, then $\int_a^b f(x)dx = \lim_{M\to b^-} \int_a^M f(x)dx$.

- If f is continuous on $\{[a,c)\cup(c,b]\}$ & has infinite discontinuity at c, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

- These improper integrals are said to be convergent if the limit is finite, and divergent otherwise.

e.g: Evaluate the following improper integrals.

1) $\int_{-\infty}^{\infty} \frac{x}{(1+x^2)^2} dx$

2) $\int_1^{\infty} \frac{x^2}{(1+x^2)^2} dx$

3) $\int_3^{\infty} \frac{1}{x(\ln x)^2} dx$

4) $\int_1^{\infty} \frac{1}{\sqrt{x}(\sqrt{x}+1)} dx$

5) $\int_{-1}^{0} \frac{1}{x^2} dx$

6) $\int_1^2 \frac{1}{(x-1)^2} dx$

7) $\int_0^2 \frac{1}{(x-1)^2} dx$

8) $\int_0^{\infty} \cos x\, dx$

9) $\int_0^{\pi/2} \sec^2 x\, dx$

10) $\int_1^{\infty} \left(\frac{1}{x^3}+\frac{1}{x^2}\right) dx$.

Applications of Integration

* Area of a region in the plane

If function f is continuous & nonnegative on an interval $[a,b]$, then area A of the region bounded by $x=a$, $x=b$, the x-axis & graph of f is

$$A = \int_a^b f(x)\,dx.$$

Note:

(i) If function f is negative on $[a,b]$, then

$$\text{area } A = \int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx.$$

Thus, to find area A bounded by graph of f & the x-axis:

(a) Determine subintervals where $f(x) \geq 0$ & $f(x) < 0$.

(b) Integrate f over those subintervals on which $f(x) \geq 0$.

(c) Integrate $-f$ over those subintervals on which $f(x) < 0$.

(d) The total area is the sum of the values computed in (b) & (c).

(ii) Area A of the region bounded by graphs of two continuous functions f & g with $f(x) \geq g(x)$ on the interval $[a,b]$ is

$$A = \int_a^b (f(x) - g(x))\,dx.$$

e.g.:

1) Find area of the region bounded by the x-axis,

(a) lines $x=2, x=4$ and graph of $f(x) = x^2 - 2x$.

(b) lines $x=0, x=4$ and graph of $f(x) = x^2 - 2x$.

(c) lines $x=0$, $x=3$ and graph of $g(x) = |x-2|$.

(d) and graph of $h(x) = x^2 - 3x + 2$ on the interval $[0,3]$.

(e) line $x=2$ & graph of $k(x) = x^3 - 2x^2 + 3$.

2) Find area of the region bounded by graphs of

(a) $f(x) = \sqrt{x}$ & $g(x) = -x$ on the interval $[1,4]$.

(b) $h(x) = \sqrt{x}$ & $k(x) = x^2$

(c) $g(x) = x^3 - 2x^2$ & $h(x) = x^2 - 2x$

(d) $y^2 = x+1$ & $x+y=1$.

* **Volume of a solid with known cross section**

Suppose S is a solid lying between planes perpendicular to the x-axis that pass through $x=a$ & $x=b$.

If $A(x)$ represents cross-sectional area of S perpendicular to the x-axis which is continuous on the interval $[a,b]$, then volume V of S is

$$V = \int_a^b A(x)\,dx.$$

* **Volume of Revolution**

If function f is continuous & nonnegative on the interval $[a,b]$, then volume V of the solid generated by revolving graph of f about the x-axis on $[a,b]$ is $V = \pi\int_a^b (f(x))^2\,dx$.

• If functions f & g are continuous & nonnegative on the interval $[a,b]$ with $f(x) \geq g(x)$, then volume V of the solid generated by revolving graphs of f & g about the x-axis on $[a,b]$ is

$$V = \pi\int_a^b \left((f(x))^2 - (g(x))^2 \right) dx.$$

e.g.:

1) Let S be a cone lying between planes perpendicular to the x-axis that pass through $x=0$ & $x=6$, and having circular cross sections perpendicular to the x-axis with radius $\frac{x}{2}$, for each $x \in [0,6]$. Find volume of S.

2) Base of solid S is bounded on the xy-plane by graphs of $x=y^2$ & $x=4$. Each cross section perpendicular to the x-axis is an isosceles triangle of height 1. Find volume of S.

3) Find volume of a solid generated by revolving graph of $f(x)=\sqrt{x}$

(a) about the x-axis on the interval $[1,4]$.

(b) about the y-axis on the interval $[1,4]$.

4) Find volume of a solid generated by revolving graphs of $f(x)=x+1$ & $g(x)=x^3+1$ about the x-axis in the first quadrant.

* **Arc Length (or Length of a curve on an interval)**

If function f is continuous on an interval $[a,b]$ & differentiable on (a,b), then arc length L of graph of f on $[a,b]$ is

$$L = \int_a^b \sqrt{1+(f'(x))^2}\, dx.$$

* **Surface Area of Revolution**

If function f is continuous & nonnegative on an interval $[a,b]$, and differentiable on (a,b), then surface area S of the solid generated by revolving graph of f about the x-axis on $[a,b]$ is

$$S = 2\pi \int_a^b f(x)\cdot\sqrt{1+(f'(x))^2}\, dx.$$

e.g:

1) Find arc length of graph of

(a) $f(x) = x^{3/2}$ on the interval $[1, 4]$.

(b) $g(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $[1, 2]$.

(c) $h(x) = x^2$ on the interval $[0, \frac{1}{2}]$.

2) Find area of the surface generated by revolving graph of function $f(x) = \dfrac{x^3}{3}$ about the x-axis on the interval $[0, 2]$.

3) Show that surface area of a sphere of radius r is $4\pi r^2$.